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Holography for non-critical superstrings

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Abstract: We argue that a class of "non-critical superstring" vacua is holographically related to the (non-gravitational) theory obtained by studying string theory on a singular Calabi-Yau manifold in the decoupling limit $g_s \to 0$. In two dimensions, adding fundamental strings at the singularity of the CY manifold leads to conformal field theories dual to a recently constructed class of AdS_3 vacua. In four dimensions, special cases of the construction correspond to the theory on an $NS5$-brane wrapped around a Riemann surface.

Keywords: Superstrings and Heterotic Strings, String Duality, Superstring Vacua.

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Contents

1. Introduction

In [1] it was proposed that vacua of string theory which asymptote at weak coupling to linear dilaton backgrounds are holographic – string theory in such vacua is dual to a theory without gravity in a lower dimension. The dual theory is in general not a local QFT.

The only example discussed in detail in [1] was string theory in the near-horizon geometry of $NS5$-branes [2], which was argued to be dual to the non-local theory that governs the dynamics of $NS5$-branes at vanishing string coupling [3] (see also [4, 5]). It was further noted in [1] that a rich set of linear dilaton vacua is provided by the "non-critical superstring" construction of [6]. The problem of identifying the corresponding non-gravitational spacetime theory was (in general) left open in [1].

The main purpose of this note is to fill this gap and propose dual descriptions for a large class of non-critical superstring models in d spacetime dimensions.[1] We will argue that the dual theory is obtained by studying string theory on

$$\mathbb{R}^{d-1,1} \times X^{2n}\,, \qquad 2n = 10 - d\,, \tag{1.1}$$

[1] We will consider the cases $d = 2, 4, 6$.

where X^{2n} is a singular Calabi-Yau manifold. Sending $g_s \to 0$ at fixed l_s gives rise as in [3] to a d dimensional theory without gravity describing the dynamics of modes living near the singularity on X^{2n}. This theory is dual to string theory in a background which approaches at weak coupling $\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times \mathcal{N}$, where $\mathbb{R}_\phi$ is the real line along which the dilaton changes linearly, and $\mathcal{N}$ is related to X^{2n} in a way described below.

In two dimensions, the construction involves string theory on $\mathbb{R}^{1,1} \times X^8$, where X^8 is a singular CY fourfold. Adding fundamental strings at the singularity in X^8 and flowing to the infrared leads to a two dimensional CFT which is dual via the AdS/CFT correspondence [7] to an $N = 2$ supersymmetric AdS_3 vacuum of the sort recently discussed in [8, 9] following [10].

In four dimensions, special cases of the construction correspond to the theory on an $NS5$-brane with worldvolume $\mathbb{R}^{3,1} \times \Sigma$, where Σ is a Riemann surface; these examples might be of interest for describing QCD via branes [11, 12].

The plan of this paper is the following. We start in section 2 with a review of the non-critical superstring construction of [6]. We discuss the symmetry structure of these string vacua and construct some observables which belong to short representations of spacetime supersymmetry.

In section 3 we define the dual theories (1.1) and study some of their properties. We propose the duality and perform a few simple checks of its validity. We also point out the similarity of our proposal to the "duality" between large N matrix quantum mechanics in the double scaling limit and $1 + 1$ dimensional string theory in a linear dilaton background [13].

In section 4 we discuss in turn two, four and six dimensional examples of our construction. We make contact with recent work on string theory on AdS_3, and with theories on branes which are relevant for solving strongly coupled gauge theories. We also discuss the resolution of strong coupling singularities of non-critical superstrings.

Some of the technical details are contained in an appendix.

2. Non-critical superstrings

Consider superstring propagation on

$$\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times \mathcal{N}\,, \tag{2.1}$$

where $\mathcal{N}$ is a manifold whose properties will be specified below. The real line $\mathbb{R}_\phi$ is parametrized by ϕ. The dilaton Φ is linear in ϕ,

$$\Phi = -\frac{Q}{2}\phi\,. \tag{2.2}$$

The string coupling $g_s \sim e^\Phi$ goes to zero as $\phi \to \infty$ and diverges as $\phi \to -\infty$. In regions where g_s is large, the perturbative definition of the theory is not useful. To fully define string theory in the background (2.1), (2.2), one must either eliminate the strong coupling region, or provide a definition of the theory at strong coupling.

In the rest of this section we summarize some properties of the worldsheet description of the vacuum (2.1); this description is reliable in regions where the string coupling is small. The linear dilaton (2.2) implies that the worldsheet stress-tensor and central charge of ϕ are

$$\begin{aligned} T_\phi(z) &= -\frac{1}{2}\left[(\partial_z\phi)^2 + Q\partial_z^2\phi\right], \\ c_\phi &= 1 + 3Q^2 . \end{aligned} \tag{2.3}$$

The worldsheet superpartner of ϕ is ψ^ϕ; the superconformal current is

$$T_F = \psi^\phi\partial_z\phi + Q\partial_z\psi^\phi . \tag{2.4}$$

Consistency of fermionic string propagation requires the worldsheet theory on $\mathcal{N}$ to be an $N=1$ SCFT with central charge $c_{\mathcal{N}} = 3(n - 1/2 - Q^2)$, where n is given in (1.1). We will also assume that $\mathcal{N}$ is compact and non-singular, so that the worldsheet SCFT on $\mathcal{N}$ is unitary and has a discrete spectrum of scaling dimensions.

The construction of [6] requires that the worldsheet SCFT on $\mathcal{N}$ have the following additional properties:

(a) An affine U(1) symmetry with supercurrent

$$\psi^{\mathrm{U}(1)} + \theta J^{\mathrm{U}(1)} . \tag{2.5}$$

(b) The coset $\mathcal{N}/\,\mathrm{U}(1)$ must have an $N=2$ superconformal symmetry with central charge

$$c^{\mathcal{N}/\,\mathrm{U}(1)} = 3(n - 1 - Q^2) . \tag{2.6}$$

If both of these conditions hold, one can construct a type II string vacuum with (at least) $2^{\frac{d}{2}+1}$ supercharges as follows. It is convenient to write the affine current (2.5) as

$$J^{\mathrm{U}(1)} = i\partial Y , \tag{2.7}$$

where Y is a canonically normalized scalar: $Y(z)Y(w) \sim -\log(z-w)$. The $N=2$ superconformal algebra on $\mathcal{N}/\,\mathrm{U}(1)$ contains a $\mathrm{U}(1)_R$ current $J_R^{\mathcal{N}/\mathrm{U}(1)}$ normalized as

$$J_R^{\mathcal{N}/\mathrm{U}(1)}(z) J_R^{\mathcal{N}/\mathrm{U}(1)}(w) \sim \frac{1}{3}\frac{c^{\mathcal{N}/\mathrm{U}(1)}}{(z-w)^2} , \tag{2.8}$$

which can be expressed in terms of a canonically normalized scalar Z as

$$J_R^{\mathcal{N}/\mathrm{U}(1)} = i\sqrt{\frac{c^{\mathcal{N}/\mathrm{U}(1)}}{3}}\partial Z \equiv ia\partial Z , \qquad a \equiv \sqrt{n-1-Q^2} . \tag{2.9}$$

It is also convenient to bosonize the worldsheet fermions ψ^ϕ, $\psi^{\mathrm{U}(1)}$ in terms of a scalar field H: $\partial H = \psi^\phi \psi^{\mathrm{U}(1)}$. The spacetime supercharges are given by [6]

$$
\begin{aligned}
Q^+_\alpha &= \oint dz\, e^{-\frac{\varphi}{2}} e^{-\frac{i}{2}(H + aZ - QY)} S_\alpha\,, \\
Q^-_{\bar\alpha} &= \oint dz\, e^{-\frac{\varphi}{2}} e^{\frac{i}{2}(H + aZ - QY)} S_{\bar\alpha}\,,
\end{aligned}
\tag{2.10}
$$

where φ is the scalar field arising in the bosonized β, γ superghost system of the fermionic string, and S_α, $S_{\bar\alpha}$ are spinors of $\mathrm{SO}(d-1,1)$ [14]. For $d = 2 \mod 4$, S_α, $S_{\bar\alpha}$ are isomorphic spinors, while for $d = 0 \mod 4$ they are distinct. It is not difficult to check that the supercharges (2.10) are BRST invariant and mutually local on the worldsheet (and thus physical), and that they form the spacetime superalgebra ($\mu = 0, 1, \ldots, d-1$)

$$
\{Q^+, Q^-\} = \gamma^\mu P_\mu\,. \tag{2.11}
$$

All the other anticommutators vanish. In (2.11) P_μ is the momentum along $\mathbb{R}^{d-1,1}$ and γ^μ are the corresponding Dirac matrices. The supercharges (2.10) carry charge $\pm Q/2$ under the U(1) symmetry (2.7), which is thus an R-symmetry in spacetime. It is customary to normalize the R-charge so that the supercharges have $R = \pm 1$. Thus we define

$$
R = i\frac{2}{Q} \oint \partial Y\,. \tag{2.12}
$$

In a type II string, there are also supercharges $\bar{Q}^\pm$ that arise in a similar fashion from the other worldsheet chirality. For $d = 2 \mod 4$, $\bar{Q}^+$ and $\bar{Q}^-$ transform in isomorphic spinor representations of $\mathrm{SO}(d-1,1)$. If furthermore $Q^\pm$ and $\bar{Q}^\pm$ transform as isomorphic spinors, the theory has chiral supersymmetry in spacetime; if $Q^\pm$ and $\bar{Q}^\pm$ transform as different spinors, the spacetime supersymmetry is non-chiral. This is analogous to the choice of IIA or IIB strings in ten dimensions. The full R-charge for the type II case is $R + \bar{R}$, where $\bar{R}$ is the antiholomorphic analog of (2.12).

Observables in linear dilaton theories correspond to non-normalizable vertex operators whose wavefunctions diverge in the weak coupling region $\phi \to \infty$ [15]. We next discuss a few examples. Consider an $N = 1$ superconformal primary in the CFT on $\mathcal{N}$ which is primary with charge q under the $\mathrm{U}(1)_R$ symmetry (2.7), and whose projection in $\mathcal{N}/\,\mathrm{U}(1)$, V, is primary under the full $N = 2$ worldsheet superconformal algebra. Such an operator can be written as

$$
e^{iqY} V\,. \tag{2.13}
$$

We denote the scaling dimension of V by Δ_V and its $\mathrm{U}(1)_R$ charge (under (2.9)) by Q_V. Unitarity of the $N = 2$ SCFT on $\mathcal{N}/\,\mathrm{U}(1)$ implies that

$$
\Delta_V \geq \frac{|Q_V|}{2}\,, \tag{2.14}
$$

with equality when V is a chiral operator in the worldsheet theory on $\mathcal{N}/\,\mathrm{U}(1)$. One can form a physical observable out of (2.13) as follows:

$$e^{-\varphi-\bar{\varphi}}e^{i\vec{k}\cdot\vec{x}+iqY+\beta\phi}V\,, \tag{2.15}$$

where $\vec{k}$ is the momentum along $\mathbb{R}^{d-1,1}$ and β is the "Liouville dressing." The mass-shell condition and GSO projection (mutual locality of (2.15) with (2.10)) lead to the following physical state constraints:

$$\frac{1}{2}|\vec{k}|^2+\frac{1}{2}q^2-\frac{1}{2}\beta(\beta+Q)+\Delta_V=\frac{1}{2}\,,\qquad Q_V-qQ\in 2\mathbb{Z}+1\,. \tag{2.16}$$

Non-normalizability of the wavefunction as $\phi\to\infty$ implies that (2.15) must satisfy[2]

$$\beta\geq-\frac{Q}{2}\,. \tag{2.17}$$

One can think of (2.15) as a fermionic string "tachyon";[3] because of the chiral GSO projection it is of course never tachyonic [6].

Another set of observables corresponds to "gravitons," whose $(-1,-1)$ picture vertex operators have the form:

$$e^{-\varphi-\bar{\varphi}}\xi_{\mu\nu}\psi^\mu\bar{\psi}^\nu e^{i\vec{k}\cdot\vec{x}+iqY+\beta\phi}V\,, \tag{2.18}$$

where ψ^μ are worldsheet fermions on $\mathbb{R}^{d-1,1}$, $\xi_{\mu\nu}$ is the polarization, and the physical state conditions are[4]

$$\frac{1}{2}|\vec{k}|^2+\frac{1}{2}q^2-\frac{1}{2}\beta(\beta+Q)+\Delta_V=0\,,\qquad Q_V-qQ\in 2\mathbb{Z}\,. \tag{2.19}$$

As in critical string theory, there is an infinite tower of observables generalizing (2.15) and (2.18); we will not discuss them here.

Chiral operators, that belong to short representations of spacetime supersymmetry, are of special interest. Such operators can be obtained by taking V to be a chiral operator on the worldsheet, for which the inequality (2.14) is saturated (we will take Q_V to be non-negative, and $\Delta_V=Q_V/2$), and setting the momentum along $\mathbb{R}^{d-1,1}$ to zero. Consider first the "tachyon" (2.15). One can solve the constraints (2.16) by setting $\beta=q$, $Q_V-qQ=1$. It can be shown that the resulting operator belongs to a short representation of spacetime supersymmetry. The R-charge (2.12) of a chiral tachyon operator (2.15) is

$$R_V=\frac{2q}{Q}=\frac{2(Q_V-1)}{Q^2}\,. \tag{2.20}$$

[2]For $\beta=-Q/2$ the non-normalizable solution is $\phi\exp(-Q\phi/2)$.

[3]It is a tachyon if V is a function on $\mathcal{N}/\,\mathrm{U}(1)$. In general, (2.15) corresponds to an excited state of the string.

[4]In addition, there are transversality conditions on $\xi_{\mu\nu}$ which we will not specify.

Note that the constraint (2.17) applied to these chiral operators implies[5] that $Q_V - 1 > -Q^2/2$ or

$$Q_V + \frac{Q^2}{2} - 1 > 0\,. \tag{2.21}$$

For $Q^2 > 2$ this constraint is automatically satisfied, since by assumption $Q_V \geq 0$, while for $Q^2 < 2$ it implies that some of the chiral operators on the worldsheet do not give rise to chiral operators in spacetime.

The last statement should be qualified somewhat. First, note that the chiral operators just constructed are *top components* of chiral superfields in spacetime. One way of seeing that is the following. In the $(-1,-1)$ picture, the operator (2.15) with $\beta = q$ and $\vec{k} = 0$ has the form

$$e^{-\varphi-\bar{\varphi}} e^{q(\phi+iY)} V\,; \tag{2.22}$$

$e^{q(\phi+iY)}V$ is the bottom component of a *worldsheet* chiral superfield with scaling dimension $(1/2, 1/2)$. The $(0,0)$ picture vertex operator is the top component of this chiral superfield. Such operators can be added to the worldsheet lagrangean without breaking $(2,2)$ superconformal symmetry [16], and therefore also spacetime supersymmetry.

What is the spacetime interpretation of adding the operator (2.22) to the worldsheet lagrangean? According to [1], the operator (2.15) corresponds to an off shell observable $O(\vec{x})$ in the dual, non-gravitational theory.[6] The $\vec{k} = 0$ mode of (2.15), given by (2.22), corresponds to $\int d^d\vec{x}\, O(\vec{x})$, and adding it to the worldsheet lagrangean corresponds in the dual theory to adding $\int d^d\vec{x}\, O(\vec{x})$ to the spacetime action. The fact that doing that does not break supersymmetry implies that $O(\vec{x})$ is the top component of a chiral superfield.

The bottom component of this superfield, V_{bottom}, is killed by half of the supercharges in (2.10), Q^+_α. Acting on V_{bottom} with the remaining half of the supercharges, the $2^{d/2-1}$ $Q^-_{\bar{\alpha}}$, gives rise to (2.22). Hence the R-charge of V_{bottom} is

$$R_{\text{bottom}} = \frac{2(Q_V - 1)}{Q^2} + 2^{d/2-1}\,. \tag{2.23}$$

Note that it always satisfies $R_{\text{bottom}} \geq \frac{2(Q_V-1)}{Q^2} + 1$. The constraint (2.21) implies that R_{bottom} is always positive.

Therefore, what we have found before is that chiral worldsheet operators which do not satisfy (2.21) do not give rise to observables of the form (2.15) which are *top* components of *chiral* spacetime superfields. Instead, they give rise to *bottom*

[5] In the case $\beta = -Q/2$ one does not find a chiral operator because of the insertion of ϕ in front of the exponential (see footnote 2).

[6] We are speaking loosely here. The dual theory probably does not have local observables. It does have observables labeled by arbitrary d dimensional momenta such as (2.15), (2.18), but there may be subtleties in Fourier transforming to real space (see [17, 18] for further discussion).

components of *antichiral* spacetime superfields. Indeed, returning to (2.15), another solution to the physical state conditions (2.16) with $\vec{k} = 0$ is $\beta = -q - Q$, $q = (Q_V - 1)/Q$. Since q has the same value as before, the R-charge of the corresponding spacetime operator,

$$e^{-\varphi-\bar{\varphi}} e^{-(q+Q)\phi+iqY} V \,, \tag{2.24}$$

is again given by (2.20). The condition (2.17) for non-normalizability of (2.24) is

$$Q_V + \frac{Q^2}{2} - 1 < 0 \,, \tag{2.25}$$

the opposite of (2.21). The R-charge (2.20) is negative definite in the regime (2.25), which is simply the statement that the corresponding operator is the bottom component of an antichiral spacetime superfield.

The top components of spacetime superfields (2.22), (2.24) play an important role in the theory. As discussed in [6, 1], due to the linear dilaton (2.2) the string coupling diverges as $\phi \to -\infty$. There are different known mechanisms for regulating this strong coupling divergence. One that was used in [6], and will play a role below, is to add to the worldsheet lagrangean an operator of the form (2.22) with $-Q/2 < q < 0$. This sets a scale for fluctuations of ϕ. For $q < 0$ the vertex operator (2.22) grows as $\phi \to -\infty$ (the wavefunction, which differs from the vertex operator by a factor of g_s, is always supported at $\phi \to \infty$). Thus, it prevents the system from running to the strong coupling region $\phi \to -\infty$. Vertex operators which grow as $\phi \to -\infty$ can be thought of as relevant operators in linear dilaton backgrounds. Ref. [6] used the operator (2.22) with $V = 1$, the "$N = 2$ cosmological constant" of $N = 2$ Liouville theory, to set the scale. This operator exists only for $Q^2 > 2$ (2.21). For $Q^2 < 2$ one may try to use a relevant worldsheet chiral operator with $1 > Q_V > 1 - Q^2/2$, if one exists.

Alternatively, one may use the top component of an antichiral superfield associated with (2.24). However, in this case there is the following subtlety. One nice property of (2.22) is that if the operator V is the bottom component of a relevant superfield in the worldsheet SCFT on $\mathcal{N}/\mathrm{U}(1)$ (i.e. $Q_V < 1$), the dressed operator (2.22) is relevant in spacetime (i.e. $q < 0$), and vice versa. For the antichiral operators (2.24), the relation is more complicated. For such operators

$$\beta = -q - Q = \frac{1 - Q_V - Q^2}{Q} \,. \tag{2.26}$$

They are relevant in spacetime (i.e. have $\beta < 0$) if

$$1 - \frac{Q^2}{2} > Q_V > 1 - Q^2 \tag{2.27}$$

(the first inequality is (2.25)). As mentioned above, for $Q^2 > 2$ (2.27) has no solutions with $Q_V \geq 0$. For $2 > Q^2 > 1$ all operators (2.24) which satisfy (2.25) are relevant in spacetime (as well as on the worldsheet). For $Q^2 < 1$, operators with $Q_V < 1 - Q^2$ are relevant on the worldsheet but irrelevant in spacetime.

Comments:

1. Our discussion of the relevance in spacetime concerned the operators (2.24), which are bottom components of spacetime superfields. Since the spacetime supercharges do not change the ϕ dependence of the wavefunctions, the same analysis holds for the top components, which are the operators one would actually add to the worldsheet lagrangean.

2. Equation (2.26) has the strange property that the more relevant the operator V is on the worldsheet, the less relevant the corresponding operator (2.24) is in spacetime. We do not understand this behavior.

3. A similar analysis can be performed for gravitons (2.18). Chiral operators correspond to $\beta = q = Q_V/Q$. The corresponding R-charge is

$$R_V = \frac{2Q_V}{Q^2}\,. \tag{2.28}$$

To recapitulate, the procedure outlined above leads to a non-critical superstring theory in d dimensions with $2^{d/2+1}$ supercharges. The theory is not conformal, and the arguments of [1] suggest that it is holographically related to a (perhaps non-local) d dimensional theory without gravity.

In the next section we will propose a candidate for the theory without gravity which is related by the duality of [1] to the string vacua described above. We will specialize[7] to a class of backgrounds (2.1) for which the worldsheet CFT on $\mathcal{N}$ is a product of the S^1 (2.5) and a Landau-Ginzburg (LG) $N = 2$ SCFT of $n + 1$ chiral superfields z_a, $a = 1, \ldots, n + 1$, with superpotential

$$W(z_a) = F(z_a)\,, \tag{2.29}$$

where F is a quasi-homogenous polynomial with weight one under $z_a \to \lambda^{r_a} z_a$, i.e.

$$F(\lambda^{r_a} z_a) = \lambda F(z_a)\,, \qquad \lambda \in \mathbb{C}\,, \tag{2.30}$$

for some set of positive weights r_a. Here and below we take F to be transverse, i.e. the only point at which all derivatives $\partial_{z_a} F$ vanish is the origin, $z_a = 0$.

In applications, the worldsheet CFT on $\mathcal{N}$ is in general not a *direct* product of S^1 and the LG model (2.29). Often, one can reach a point in moduli space where the two are decoupled, and we will assume that we are at such a point in the analysis below. It is easy to generalize to situations where this is not the case.

[7] It is probably possible to generalize our discussion to the most general compact manifold $\mathcal{N}$ satisfying the constraints described above. This is left for future work.

The worldsheet central charge c_W corresponding to (2.29) is

$$\frac{1}{3}c_W = \sum_{a=1}^{n+1}(1-2r_a) = n+1-2\sum_a r_a\,. \tag{2.31}$$

It is useful to define

$$r_\Omega \equiv \sum_{a=1}^{n+1} r_a - 1\,, \tag{2.32}$$

in terms of which

$$c_W = 3(n-1-2r_\Omega)\,. \tag{2.33}$$

Comparing (2.33) to (2.6) we see that

$$Q^2 = 2r_\Omega\,. \tag{2.34}$$

In particular, r_Ω must be positive in this construction.

Chiral operators are constructed as in the general discussion above. The worldsheet chiral operators V are in this case polynomials[8] in z_a, $A_i(z_a)$, which have weights r_i under $z_a \to \lambda^{r_a} z_a$. Their worldsheet R-charges Q_V are equal to r_i. As discussed above, one can construct chiral operators in spacetime out of the A_i by dressing them in different ways. The dressing (2.15) gives rise to the chiral operator

$$e^{-\varphi-\bar{\varphi}} e^{q(\phi+iY)} A_i\,, \tag{2.35}$$

where $q = (r_i-1)/Q$; the R-charge (2.20) is

$$R_i = \frac{r_i - 1}{r_\Omega}\,. \tag{2.36}$$

As in the general discussion, for $r_\Omega < 1$ not all homogenous polynomials A_i give rise by using (2.35) to top components of chiral superfields in the spacetime theory. The constraint (2.21) implies that only those with

$$r_i + r_\Omega - 1 > 0 \tag{2.37}$$

give rise to such operators. The rest of the A_i give bottom components of antichiral superfields.

Similarly, (2.18) gives rise to a chiral operator with R-charge r_i/r_Ω (see (2.28)).

[8] A_i are arbitrary quasi-homogenous polynomials, defined modulo the equations of motion, $\partial_{z_a} F(z_a) = 0$.

3. The dual theory

3.1 The dynamics near a singularity

Consider type II string theory on the manifold $\mathbb{R}^{d-1,1} \times \bar{X}^{2n}$, where $\bar{X}^{2n}$ is a CY n-fold. If $\bar{X}^{2n}$ is smooth, the theory becomes free in the limit $g_s \to 0$. If however $\bar{X}^{2n}$ contains an isolated singular point, y_0, one expects in general to find in this limit a theory with a scale l_s describing modes localized in the vicinity of y_0. To study the dynamics near the singularity, it is sufficient to consider the part of the original compact manifold which is close to y_0, and to replace $\bar{X}^{2n}$ by an appropriate non-compact manifold X^{2n} with the same singularity. The non-compactness of X^{2n} also allows one to consider singularities that cannot be embedded in a compact CY manifold $\bar{X}^{2n}$.

A well-known example with $d = 6$ is obtained by taking $\bar{X}^{2n}$ to be a $K3$ manifold with an ADE singularity. In the vicinity of the singular point, the $K3$ can be replaced by an appropriate non-compact ALE space. In the limit $g_s \to 0$ bulk string theory decouples, and one is left with a non-local theory without gravity with a scale l_s [3].[9] While only ADE singularities of some fixed finite order can be embedded in a compact $K3$, the non-compactness of an ALE space allows one to consider all ADE singularities.

We next list a few properties of these decoupled theories. We focus on the special case of quasi-homogenous hypersurface singularities (see [23] for a recent discussion), although it is probably possible to generalize the discussion to other cases. Thus, we take the non-compact CY manifold X^{2n} describing the vicinity of the singularity to be the hypersurface $F(z_1, \ldots, z_{n+1}) = 0$ in $\mathbb{C}^{n+1}$, where F is a polynomial which transforms as $F \to \lambda F$ under

$$z_a \to \lambda^{r_a} z_a\,, \tag{3.1}$$

as in (2.30). The point $z_a = 0$ is a singular point in X^{2n} and one expects to find a decoupled theory living in its vicinity in the limit $g_s \to 0$. Isometries of X^{2n} give rise to symmetries of this theory. In particular, the hypersurface $F(z_a) = 0$ has a U(1) symmetry which acts as (3.1) with $|\lambda| = 1$. The holomorphic n-form Ω on X^{2n},

$$\Omega = \frac{dz_1 \wedge \ldots \wedge dz_n}{\partial F / \partial z_{n+1}}\,, \tag{3.2}$$

has charge $r_\Omega = \sum_a r_a - 1$ under this symmetry. If $r_\Omega \neq 0$, this U(1) symmetry is an R-symmetry. Indeed, since one can write Ω in terms of a covariantly constant spinor η on X^{2n} as $\Omega_{i_1 \cdots i_n} = \eta^t \Gamma_{i_1 \cdots i_n} \eta$, the R-charge of the spacetime supercharges is $\pm 1/2$ that of Ω, i.e. $\pm r_\Omega/2$. Moreover, for any hypersurface singularity which occurs at a finite distance in CY moduli space, r_Ω must be positive [23]. Since we are interested primarily in singularities that occur at a finite distance, we will restrict to the case $r_\Omega > 0$ below.

[9] In fact, [3] considered the decoupling limit $g_s \to 0$ in string vacua containing coincident $NS5$-branes, which are related to ALE spaces by T-duality [19, 20, 21] (see also [22]).

The singular hypersurface $F(z_a) = 0$ can be deformed to

$$F(z_a) + \sum_i t_i A_i(z_a) = 0\,, \tag{3.3}$$

where $A_i(z_a)$ are quasi-homogenous polynomials with weight r_i. In the decoupled theory near the singularity, t_i correspond to couplings of top components of chiral superfields. From (3.3) we see that the $\mathrm{U}(1)_R$ charge of t_i is $1-r_i$. The corresponding operator in the spacetime theory, which we will also denote by A_i, thus has charge $r_i - 1$. If we normalize the R-charge so that the supercharges have charge ± 1, the charge of A_i becomes

$$\frac{2(r_i - 1)}{r_\Omega}\,. \tag{3.4}$$

Not all couplings t_i can be turned on in this theory. By requiring that the kinetic energy for t_i diverge as $z_a \to \infty$, so that t_i correspond to non-fluctuating couplings, [23] found that only modes that satisfy (2.37) exist in this theory.

3.2 The proposed duality

We propose that the non-gravitational theory describing the dynamics of string theory on $\mathbb{R}^{d-1,1} \times X^{2n}$ in the limit $g_s \to 0$, where X^{2n} is a non-compact CY n-fold with an isolated singular point y_0, is dual to string theory on $\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times \mathcal{N}$, where $\mathcal{N}$ is the manifold consisting of all points in X^{2n} at a fixed distance from the singular point y_0.[10]

For the case of hypersurface singularities $F(z_a) = 0$ discussed above, the manifold $\mathcal{N}$ can be thought of as $\mathcal{N} = X^{2n}/\mathbb{R}_+$ where $\mathbb{R}_+$ acts on z_a as (3.1) with $\lambda \in \mathbb{R}_+$. This quotient has a residual $\mathrm{U}(1)$ action (3.1) with $|\lambda| = 1$ which is the $\mathrm{U}(1)_R$ symmetry discussed above. $\mathcal{N}/\,\mathrm{U}(1) \simeq X^{2n}/\mathbb{C}$ is the $n-1$ complex dimensional hypersurface $F(z_a) = 0$ in the n dimensional weighted projective space $\mathbb{WCP}^n_{r_1,\ldots,r_{n+1}}$.

Is the manifold $\mathcal{N}$ described above a good background for string propagation? As discussed in section 2, for this to be the case the non-linear sigma-model on $\mathcal{N}/\,\mathrm{U}(1)$ must be $(2,2)$ superconformal. One way to think about this issue is to embed this non-linear sigma model in a gauged linear sigma-model following [24]. One studies a $(2,2)$ supersymmetric $\mathrm{U}(1)$ gauge theory with matter fields z_a ($a = 1, \ldots, n+1$) with charges r_a and an additional field P with charge -1. The gauge invariant superpotential is taken to be

$$W = PF(z_a)\,. \tag{3.5}$$

Since the gauge group is abelian, one can add to the lagrangean a Fayet-Iliopoulos D-term, r. Classically, when r is large and positive the low energy dynamics of the linear sigma model corresponds to the non-linear sigma-model on $\mathcal{N}/\,\mathrm{U}(1)$, and r

[10] More precisely, as we will see below, the string background involves the worldsheet CFT to which the sigma model on $\mathcal{N}$ flows in the IR.

can be thought of as the size of the space (a Kähler modulus). When r is large and negative one finds a LG model[11] with $W = F$ [24].

Quantum mechanically, the behavior of the theory depends on the sum of the gauge charges of the matter fields [24]. When this sum vanishes, the quantum picture is closely related to the classical one: r remains a modulus in the quantum theory (i.e. it is truly marginal), and changing this modulus between $-\infty$ and ∞ interpolates between LG theory and the non-linear sigma-model. Thus, in this case both are conformal, and provide different descriptions of a single moduli space of CFT's.

When the sum of the charges is non-zero, there are logarithmic corrections to r at one loop. Thus, there is a β-function for r and it flows as we change the scale. When the sum of the charges is positive, r decreases as we go to longer distances, and vice-versa. The detailed RG flows in this case have not been analyzed, to our knowledge.

In our case, the sum of the gauge charges is $r_\Omega = \sum_a r_a - 1$ and, as we saw before, it is positive. Thus, one expects the Kähler modulus r of the hypersurface $\mathcal{N}/\,\mathrm{U}(1)$ to decrease with increasing scale, and presumably go to $r = -\infty$ at long distances. It is then natural to expect that the infrared fixed point of the non-linear sigma-model on $\mathcal{N}/\,\mathrm{U}(1)$ is the infrared limit of the LG theory with superpotential $W = F$ discussed in section 2 (2.29) (we will denote it below by $LG(W = F)$). Therefore, the duality proposed in the beginning of this subsection involves in this case the background $\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times \mathcal{N}$, where $\mathcal{N}$ is roughly $S^1 \times LG(W = F)$. As mentioned before, the product here may not be direct, i.e. there may be a correlation between the quantum numbers of operators in S^1 and in $LG(W = F)$.

The duality proposed here is very reminiscent of the "old matrix model" [13]. There, one considers quantum mechanics of $N \times N$ hermitean matrices in a potential $V(M)$ in the large N limit. The lagrangean is

$$\mathcal{L} = \mathrm{Tr}\left(\frac{1}{2}\dot{M}^2 - V(M)\right). \tag{3.6}$$

For generic V the dynamics depends sensitively on the detailed structure of V. However, one can fine tune the couplings in V and approach a codimension one surface in coupling space along which the dynamics described by (3.6) is singular. The physics near this singularity is universal (i.e. independent of the detailed structure of V), and for the purpose of studying it one can replace V by an inverted harmonic oscillator potential.

In the double scaling limit N is taken to infinity while approaching the singular surface in coupling space, keeping a dimensionful parameter which measures the distance from the singularity fixed. This limit leads to a non-trivial theory. Green functions of $\mathrm{U}(N)$ invariant observables have non-trivial $1/N$ expansions and one

[11] More precisely, this is an orbifold of the above LG model, which here is implemented by the GSO projection.

JHEP10(1999)035

can attempt to study non-perturbative effects in $1/N$ as well. As described in [13], these Green functions have an alternative description as S-matrix elements in a $1+1$ dimensional string theory of a scalar field and the Liouville field ϕ. The $1/N$ expansion in the double scaled matrix model is equivalent to the genus expansion of this string theory. The dilaton is linear in ϕ, and the dimensionful parameter which measures the distance (in coupling space) of (3.6) from the singularity becomes the worldsheet cosmological constant.

Our construction is very similar. String theory on $\mathbb{R}^{d-1,1} \times \bar{X}^{2n}$ is the analog of the matrix QM (3.6). In particular, it contains a lot of "non-universal" information. Approaching a point in CY moduli space where $\bar{X}^{2n}$ develops a singularity is analogous to tuning the couplings in the potential V to the singular surface. Replacing $\bar{X}^{2n}$ by X^{2n} is the analog of replacing a general V near the critical surface by the inverted harmonic oscillator potential. The limit $g_s \to 0$ in string theory on $\mathbb{R}^{d-1,1} \times X^{2n}$ is the analog of the limit $N \to \infty$ in the matrix model. Finally, the non-critical superstring on $\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times \mathcal{N}$ (2.1) is the analog of $1+1$ dimensional string theory with a linear dilaton.

The analog of turning on the worldsheet cosmological constant in our case should be taking the limit $g_s \to 0$ for a slightly resolved singularity, as in (3.3). Indeed, we will see below that in some cases one can deform the singularity from $F(z_a) = 0$ to $F(z_a) = \epsilon$, and ϵ becomes the non-critical superstring "cosmological constant" discussed in [6], which prevents the system from running to strong coupling.

We next list a few simple checks of the proposed duality:

1. The symmetry structure seems to agree. Both models have $2^{\frac{d}{2}+1}$ supercharges and a $\mathrm{U}(1)_R$ symmetry.

2. The parameter r_Ω defined by (2.32) plays an important role in both models, and must be positive in both. In string theory on $\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times S^1 \times LG(W = F)$ this is due to (2.34), while in $\mathbb{R}^{d-1,1} \times X^{2n}$ it is due (for example) to the requirement that the singularity be at a finite distance in moduli space of CY manifolds [23]. In both theories the physics changes at $r_\Omega = 1$ ($Q^2 = 2$).

3. Agreement of the chiral rings: we saw that deformations A_i of the polynomial F (3.3) give rise to chiral superfields whose top components have R-charge $2(r_i - 1)/r_\Omega$. In the theory near the singularity on X^{2n} the R-charge is given by (3.4), while in the linear dilaton vacuum it is eq. (2.36).[12] Furthermore, in both approaches it was found that only deformations that satisfy (2.37) are allowed. In both descriptions, the origin of this constraint is the requirement that the corresponding coupling be non-normalizable as $\phi \to \infty$.

[12]The R-charge given in (2.36) appears to be too small by a factor of two. The total R-charge receives a contribution from the other worldsheet chirality. The operators (2.35) are left-right symmetric; hence, the total R-charge found in section 2 is $2(r_i - 1)/r_\Omega$, in agreement with (3.4).

4. Some special cases

4.1 Two dimensional models ($d = 2$)

In this case, the duality proposed in the previous section relates the following models with four supercharges in $1 + 1$ dimensions:

1. Type II string theory on

$$\mathbb{R}^{1,1} \times \mathbb{R}_\phi \times S^1 \times LG(W = F)\,, \tag{4.1}$$

where the LG superpotential is $W = F(z_1, \ldots, z_5)$.

2. Type II string theory on $\mathbb{R}^{1,1} \times X^8$ in the limit $g_s \to 0$, where X^8 is the singular CY fourfold $F(z_1, \ldots, z_5) = 0$.

For type IIA, models 1 and 2 have $(2, 2)$ supersymmetry in $1 + 1$ dimensions. For IIB they have chiral $(4, 0)$ supersymmetry.

In some cases, one can think of theory 2 as a theory on an $NS5$-brane, making the connection to [3], [1] more apparent. For example, if we take

$$F(z_1, \ldots, z_5) = H(z_1, z_2, z_3) + z_4^2 + z_5^2\,, \tag{4.2}$$

where $H(z_1, z_2, z_3)$ describes an ADE singularity or a deformation thereof, theory (2) above can be thought of as the decoupled theory on a curved $NS5$-brane whose worldvolume is $\mathbb{R}^{1,1} \times L_4$, where L_4 is the hypersurface $H(z_1, z_2, z_3) = 0$ in $\mathbb{C}^3$ [23]. This follows[13] from a straightforward generalization of the arguments of [19]. Because of the usual chirality flip between fivebranes and singular geometries (which is essentially due to T-duality), theories 1 and 2 in type IIA are related to $NS5$-branes in IIB, and vice versa.

In theory 2, one can add p fundamental strings at the singular point in X^8, $z_a = 0$. In type IIA this does not break any further supersymmetry, while in IIB it breaks $(4, 0)$ supersymmetry to $(2, 0)$. In the infrared, the resulting theory will generically approach a non-trivial $(2, 2)$ or $(2, 0)$ superconformal fixed point. It is natural to expect that the description of this fixed point in the dual theory 1 is obtained by replacing $\mathbb{R}^{1,1} \times \mathbb{R}_\phi$ in (4.1) by AdS_3, and studying string theory on

$$AdS_3 \times S^1 \times LG\,. \tag{4.3}$$

This seems to be consistent with recent work on the AdS/CFT correspondence for branes at singularities [25, 26, 27, 28, 7].

The class of vacua (4.3) was recently discussed in [8, 9] who showed that the spacetime theory indeed has $N = 2$ superconformal symmetry. The spacetime central charge of these vacua is [10, 29, 30]

$$c_{\text{spacetime}} = 6kp\,, \tag{4.4}$$

[13] We thank K. Hori and H. Ooguri for pointing this out to us.

where k is the radius of curvature of AdS_3, or equivalently the level of the SL(2) current algebra on the worldsheet of the string, and p the number of strings. Equation (4.4) is valid for all k and to leading order in $1/p$.

By comparing the worldsheet central charge of the linear dilaton vacuum (2.3) to that of AdS_3, one finds that k is related to the parameters Q, r_Ω as follows:

$$\frac{1}{k} = \frac{Q^2}{2} = r_\Omega \,. \tag{4.5}$$

In sections 2 and 3 we saw that the physics of non-critical superstrings depends on whether r_Ω is larger or smaller than one. It is interesting to note that string theory on AdS_3 also undergoes a kind of phase transition at the point $k = r_\Omega = 1$. As discussed in [31] (see also [30]), string theory on AdS_3 has a set of excitations corresponding to long strings living at the boundary of AdS_3. The energy gap for the system to emit one of these strings is finite, and it goes to zero as $k \to 1$. Excitations of long strings form a continuum above the gap. A related fact is that these long strings become "critical" at $k = 1$. For $k > 1$ their string coupling grows as one approaches the boundary of AdS_3, while for $k < 1$ it goes to zero there [31].

It would be interesting to understand better the relation between the phase transitions observed in the linear dilaton, singular CY and AdS_3 systems.

The analysis of excitations performed for the linear dilaton vacuum in section 2 can be repeated for AdS_3. For example, the analogs of the observables (2.15) in this case are

$$e^{-\varphi-\bar{\varphi}} e^{iqY} V \Phi_h \,, \tag{4.6}$$

where Φ_h is a primary of SL(2) in the spin $j = h - 1$ representation; its spacetime scaling dimension is h (see [30] for a more detailed discussion). The physical state constraints in this case are

$$\frac{1}{2}q^2 - \frac{h(h-1)}{k} + \Delta_V = \frac{1}{2} \,, \qquad Q_V - qQ \in 2\mathbb{Z} + 1 \,. \tag{4.7}$$

Chiral operators under the spacetime $N = 2$ superconformal algebra have scaling dimension h equal to one half their R-charge[14] (2.12). Thus we set

$$h = \frac{|q|}{Q} = |q|\sqrt{\frac{k}{2}} \,. \tag{4.8}$$

Plugging in (4.7) we find

$$\frac{h}{k} + \Delta_V = \frac{1}{2} \,. \tag{4.9}$$

[14] As shown in [8, 9], (2.12) is the zero mode of the spacetime U(1) current that is part of the spacetime $N = 2$ superconformal algebra.

Taking V to be a chiral operator with $\Delta_V = Q_V/2$ as before, we find that

$$h = \frac{k}{2}(1 - Q_V)\,. \tag{4.10}$$

The second equation in (4.7) together with (4.8) then implies that q is negative, and $h = -q/Q$. Thus, the operator we found is a bottom component of an antichiral superfield. In string theory on AdS_3 only operators Φ_h with $h > 1/2$ exist.[15] Imposing this in (4.10) leads to the constraint on charges

$$Q_V + \frac{1}{k} - 1 < 0\,, \tag{4.11}$$

which is nothing but the constraint (2.25) found for the corresponding operators in the linear dilaton background.

To find analogs of (2.22) in AdS_3, we require that the bottom component of the spacetime superfield whose top component corresponds to (4.6) be chiral; this implies

$$h - \frac{1}{2} = \frac{1}{2}\left(\frac{2q}{Q} + 1\right) \Rightarrow h - 1 = \frac{q}{Q}\,. \tag{4.12}$$

Plugging in (4.7) leads to

$$h = 1 + \frac{k}{2}(Q_V - 1)\,. \tag{4.13}$$

This is the dimension of the top component of the superfield. The dimension of the bottom component (which corresponds to a RR vertex operator) is

$$h_{\text{bottom}} = \frac{1}{2} + \frac{k}{2}(Q_V - 1)\,, \tag{4.14}$$

and the constraint that in (4.13) $h > 1/2$, or equivalently that $h_{\text{bottom}} > 0$,

$$Q_V + \frac{1}{k} - 1 > 0\,, \tag{4.15}$$

is the same as the constraint (2.21) satisfied by (2.22).

Therefore, we see that there is a nice correspondence between the spectrum of operators in the non-conformal linear dilaton background (4.1), and its conformal low energy limit (4.3). In fact, as we discuss in the appendix, one can construct string backgrounds which interpolate between a linear dilaton vacuum for $\phi \to \infty$ and AdS_3 for $\phi \to -\infty$, which makes this correspondence natural.

We conclude this subsection with an example. Take

$$F(z_1, \ldots, z_5) = z_1^n + z_2^2 + z_3^2 + z_4^2 + z_5^2\,, \tag{4.16}$$

[15] The scaling dimension of Φ_h and Φ_{1-h} is $-h(h-1)/k$. Φ_h with $h > 1/2$ is physical, while the other solution is related to it by an integral transform [32], whose convergence requires that $h > 1/2$. An alternative way to see that only $h > 1/2$ gives rise to non-normalizable observables is to study the behavior of the wavefunction Φ_h at the boundary of AdS_3, as done in [30].

which corresponds to an A_{n-1} singularity in (4.2). The weights r_a are: $r_1 = 1/n$, $r_2 = r_3 = r_4 = r_5 = 1/2$. Hence, r_Ω (2.32) is given by

$$r_\Omega = 1 + \frac{1}{n} = \frac{n+1}{n}\,. \tag{4.17}$$

Note that in this case r_Ω is always bigger than one, so the subtleties of $r_\Omega < 1$ discussed in sections 2 and 3 are never encountered. The spacetime central charge (4.4) is

$$c_{\text{spacetime}} = 6\left(1 - \frac{1}{n+1}\right) p\,. \tag{4.18}$$

The chiral operators A_i (3.3) corresponding to resolutions of the singularity are $A_i = z_1^i$, $i = 0, 1, 2, \ldots, n-2$. Their $\mathrm{U}(1)_R$ charges are $Q_{A_i} = i/n$. Plugging into (4.14) we find chiral operators with spacetime scaling dimensions

$$h_i = \frac{i+1}{2(n+1)}\,, \qquad i = 0, 1, \ldots, n-2\,. \tag{4.19}$$

This spectrum looks like that corresponding to an $N = 2$ minimal model with superpotential Φ^{n+1}. The central charge (4.18) is also suggestive of that. Indeed, if we could trust (4.18) for $p = 1$ (which is far from obvious), it would be natural to expect that in the spacetime SCFT with central charge

$$c_{\text{spacetime}} = 6\left(1 - \frac{1}{n+1}\right) = \left[3 - \frac{6}{n+1}\right] + 3\,, \tag{4.20}$$

the contribution in square brackets comes from an $N = 2$ minimal model with the above superpotential. For large p, (4.18) might describe a (deformation of a) symmetric product of such minimal models, each coupled to a $c = 3$ system. It would be interesting to understand to what extent this is true, and what is the role of the $c = 3$ piece of the spacetime CFT.

A system closely related to ours was recently discussed in [23], who however set p to zero and instead turned on some discrete RR backgrounds that are difficult to analyze in our formalism. Interestingly, in the simplest case it was found in [23] that the spacetime SCFT corresponding to (4.16) is an $N = 2$ minimal model with superpotential Φ^{n+1} and the result (4.19) for the dimensions of chiral operators was obtained as well. This is encouraging, since our analysis is reliable at large p, while that of [23] works for small p; it would be interesting to understand the relation between the two.

Returning to the linear dilaton vacuum with $p = 0$, the example (4.16) allows us to demonstrate another point that was briefly mentioned above. In the non-critical superstring construction, the string coupling diverges as $\phi \to -\infty$. In the theory of the $NS5$-brane discussed in [1] the strong coupling singularity is avoided in a way that cannot be understood in string perturbation theory. Thus, the linear dilaton description has limited utility; e.g. it cannot be used for computing correlation functions.

In our case, the situation is the following. If we are studying the singular theory, i.e. take X^8 to be the manifold $F(z_1,\ldots,z_5)=0$, where F is given by (4.16), the dual description again involves an infinite throat with infinite coupling down the throat, and perturbatively nothing stops the theory from running to $\phi\to-\infty$. The singularity is again avoided non-perturbatively. However, we can resolve the singularity $F=0$ to $F=\epsilon$, by turning on the operator $A_0=1$ in (3.3). In the non-critical superstring this corresponds to adding to the worldsheet lagrangean the operator (2.35) with $A_i=1$. This is precisely the operator used in [6] to cut off the strong coupling singularity at $\phi\to-\infty$. After the resolution, string perturbation theory is well defined, and can be used reliably to calculate correlation functions.

As discussed in section 3, all this is very reminiscent of the "old matrix model" [13]. The case of the unresolved singularity $F=0$ corresponds to vanishing cosmological constant in Liouville theory (or, equivalently, vanishing condensate of the "tachyon" field in $1+1$ dimensional string theory), which is a singular limit, at least perturbatively. The resolved system $F=\epsilon$ corresponds to finite cosmological constant and much of the intuition developed in $1+1$ dimensional string theory is applicable here.[16]

Note also that for $p\neq 0$, using the result (4.19) for the dimensions of chiral operators, we see that $A_0=1$ corresponds to an operator with scaling dimension $h_0=\frac{1}{2(n+1)}$ in the low energy spacetime CFT. Thus, the spacetime superpotential Φ^{n+1} is deformed for finite ϵ to $W=\Phi^{n+1}+\epsilon\Phi$, which indeed completely resolves the n-fold singularity at $\Phi=0$. This is consistent with the fact that the non-critical superstring with finite $N=2$ cosmological constant does not appear to describe a non-trivial spacetime CFT in the infrared.

One can also study RG flows in the boundary theory, by adding relevant deformations to F, as in (3.3). This leads to a picture like that of [33]. The general relevant perturbation of the worldsheet superpotential (4.16) is

$$F(z_1,\ldots,z_5)=z_1^n+\epsilon+\sum_{i=1}^{n-2}\lambda_i z_1^i+z_2^2+z_3^2+z_4^2+z_5^2\,; \tag{4.21}$$

ϵ sets the scale, and as in [13] it can be scaled to one. The λ_i can then be thought of as dimensionful couplings that "flow with the scale." The flow can be considered either as a function of ϵ or as a function of ϕ. For large ϕ, or equivalently large ϵ, the λ_i are effectively small and the theory approaches the one with $\lambda_i=0$. For $\phi\to-\infty$, or equivalently $\epsilon\to 0$, the λ_i grow and the system generically splits into a set of decoupled vacua with $n=2$ in (4.16). By tuning the λ_i one can reach a large collection of multicritical points (see [33] for a more detailed discussion).

[16]Of course, unlike [13], one does not expect the full string theory in non-critical superstring vacua to be exactly solvable.

4.2 Four dimensional models $(d = 4)$

The duality of section 3 relates in this case the following $N = 2$ supersymmetric theories in $3 + 1$ dimensions: type II string theory on

$$\mathbb{R}^{3,1} \times \mathbb{R}_\phi \times S^1 \times LG(W = F(z_1, \ldots, z_4)) \tag{4.22}$$

and type II string theory on

$$\mathbb{R}^{3,1} \times X^6 \tag{4.23}$$

in the limit $g_s \to 0$. X^6 is the singular CY manifold $F(z_1, \ldots, z_4) = 0$.

A simple example is

$$F = z_1^2 + z_2^2 + z_3^2 + z_4^2\,, \tag{4.24}$$

for which X^6 is the conifold [34]. $r_\Omega = 1$ in this case, and the LG factor in (4.22) degenerates.

More generally, if

$$F(z_1, \ldots, z_4) = H(z_1, z_2) + z_3^2 + z_4^2\,, \tag{4.25}$$

we can think of the theory on (4.23) as the worldvolume theory on an $NS5$-brane wrapped around the Riemann surface $H(z_1, z_2) = 0$ [35]. Such fivebranes are relevant for describing $N = 2$ SYM theories using branes [11], [12].

For example, to study $NS5$-branes wrapped around the Seiberg-Witten curve at an Argyres-Douglas point [36, 37, 38], one can choose H in (4.25) to describe an ADE singularity

$$H(z_1, z_2) = \begin{cases} z_1^n + z_2^2 & A_{n-1} \\ z_1^n + z_1 z_2^2 & D_{n+1} \\ z_1^3 + z_2^4 & E_6 \\ z_1^3 + z_1 z_2^3 & E_7 \\ z_1^3 + z_2^5 & E_8 \end{cases} \tag{4.26}$$

For $d = 4$, $n = 3$ (1.1), and plugging in (2.33) gives $c_W = 6(1 - r_\Omega)$. Since $c_W \geq 0$, this implies that $r_\Omega \leq 1$. Therefore, in these examples one typically encounters the obstructions discussed in sections 2 and 3 to turning on various perturbations that resolve the singularities.

For the A_{n-1} singularity (4.26) $r_\Omega = 1/n + 1/2$. Perturbations of the form z_1^i $(i = 0, 1, \ldots, n-2)$, with $r_i = i/n$, give rise to deformations of the singularity only when (2.37) holds, i.e. for

$$i > \frac{n}{2} - 1\,. \tag{4.27}$$

Therefore, an A_{n-1} singularity cannot be completely resolved.

A possible explanation of (4.27) in the fivebrane theory is the following. At low energy, the theory of the $NS5$-brane wrapped around the Riemann surface

$H(z_1, z_2) = 0$ flows to a four dimensional $N = 2$ SCFT [36, 37, 38]. Such theories have a global $\mathrm{U}(1)_R \times \mathrm{SU}(2)_R$ symmetry. Dimensions of chiral operators are related to the $\mathrm{U}(1)_R$ charge R and $\mathrm{SU}(2)_R$ spin I via

$$D = 2I + \frac{R}{2}\,. \tag{4.28}$$

If the $\mathrm{U}(1)_R$ symmetry which becomes part of the $N = 2$ superconformal algebra in the IR can be identified at high energies, one can use (4.28) to determine the dimensions of chiral operators.

In our case, the high energy theory has a $\mathrm{U}(1)_R$ symmetry $R + \bar{R}$ (2.12), and it is natural to expect this symmetry to become the $\mathrm{U}(1)_R$ part of the global symmetry of the IR SCFT. The deformations z_1^i discussed above have $I = 0$, hence their scaling dimensions in the low energy SCFT are $D_i = \frac{1}{2}(R_i + \bar{R}_i) = R_i$. Using (2.23) (with $Q_{z_1^i} = r_i = i/n$ and $Q^2 = 2r_\Omega = (n+2)/n$) we find that the dimension of the bottom component of the superfield z_1^i is

$$D_i = 2\frac{i+2}{n+2}\,, \qquad i = 0, 1, \ldots, n-2\,. \tag{4.29}$$

All scaling dimensions in a unitary four dimensional CFT satisfy $D \geq 1$. Imposing this requirement on (4.29) leads to (4.27) (for $i = n/2 - 1$, $D_i = 1$ and the corresponding operator is a decoupled free field).

As explained in section 2, operators z_1^i which do not satisfy (4.27) give rise via (2.24) to bottom components of antichiral superfields. Since $Q^2 = 2r_\Omega$ satisfies $1 < Q^2 < 2$, these superfields are relevant. Their top components can be added to the worldsheet lagrangean to eliminate the strong coupling singularity, however, they do not correspond to new deformations of H (4.26). One can show[17] that the top component of the antichiral superfield whose bottom component is (2.24) with $V = z_1^i$ is the complex conjugate of (2.22) with $V = z_1^{n-2-i}$.

A similar analysis can be performed for the D, E series curves in (4.26). Equation (4.29) takes in general the form

$$D_i = 2\frac{e_i + 1}{h + 2}\,, \tag{4.30}$$

where h is the dual Coxeter number of the corresponding algebra and e_i are its Dynkin exponents [36, 37, 38].

[17]More generally, for $d = 4$ one can show that the worldsheet chiral ring of $LG(W = F)$ splits into two components of equal size: operators that satisfy (2.37) and those that do not. The two groups are related by spectral flow. This implies that the size of the spacetime chiral ring is half of what one might naively expect.

4.3 Six dimensional models ($d = 6$)

In this case, the duality of section 3 relates type II string theory on

$$\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times S^1 \times LG(W = F(z_1, z_2, z_3)) \tag{4.31}$$

and type II string theory on

$$\mathbb{R}^{5,1} \times X^4 \tag{4.32}$$

in the limit $g_s \to 0$. X^4 is the singular manifold $F(z_1, z_2, z_3) = 0$. The models (4.31) and (4.32) have sixteen real supercharges. In the type IIA theory they form a non-chiral $(1,1)$ supersymmetry algebra in $5+1$ dimensions, while in type IIB one finds chiral $(2,0)$ supersymmetry.

The worldsheet central charge of the LG model in (4.31) is (2.33)

$$c_W = 3 - 6r_\Omega \,. \tag{4.33}$$

For positive r_Ω this central charge is smaller than three. All unitary $N = 2$ SCFT's with central charge $c < 3$ have been classified. They correspond to $N = 2$ minimal models and are in one to one correspondence with ADE singularities. One way of describing them is as infrared fixed points of LG models with superpotential

$$F(z_1, z_2, z_3) = H(z_1, z_2) + z_3^2 \,, \tag{4.34}$$

where $H(z_1, z_2)$ is given in eq. (4.26). The manifold X^4 (4.32) $F(z_1, z_2, z_3) = 0$ is an ALE space corresponding to the appropriate ADE singularity.

In this case our proposal reduces to that of [1], where it was argued that the decoupled theory on $\mathbb{R}^{5,1} \times$ ALE is holographically related to string theory on

$$\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times S^3 \,. \tag{4.35}$$

At first sight this appears to be different from the background (4.31), but the two are in fact related as follows.

The worldsheet CFT on S^3 is described by an SU(2) WZW model associated with the ADE singularity corresponding to X^4. The $N = 2$ minimal models $LG(W = F)$ in (4.31) can be described as the coset SCFT's SU(2)/ U(1). One can decompose SU(2) WZW theory under U(1) $\times$ SU(2)/ U(1) [39]. The GSO projection fixes the radius of the U(1) and acts as an orbifold on U(1) $\times$ SU(2)/ U(1); this orbifold is equivalent by T duality to SU(2) WZW CFT [40], in agreement with (4.35).

As an example, for the A_{n-1} singularity $F(z_1, z_2, z_3) = z_1^n + z_2^2 + z_3^2$, $r_\Omega = 1/n$. None of the relevant perturbations z_1^i ($i = 0, 1, \ldots, n-2$) satisfy (2.37). Hence, they give rise to bottom components of antichiral superfields. They are killed by the eight supercharges $Q_\alpha^-, \bar{Q}_\alpha^-$. The top components of the superfields are obtained by acting on them with the eight remaining supercharges Q_α^+, $\bar{Q}_\alpha^+$.

JHEP10(1999)035

The perturbations z_1^i are relevant on the worldsheet but none of them are relevant in spacetime. Indeed, since $Q^2 = 2r_\Omega = 2/n$, the condition (2.27) implies that $Q_{z_1^i} = i/n > 1 - 2/n$, or $i > n - 2$, outside the range of available perturbations. Thus, we recover the conclusion of [2, 1] that in this case there are no relevant deformations that can be turned on that eliminate the strong coupling singularity at $\phi = -\infty$ perturbatively.

It should be mentioned for completeness that there is in fact a known way to eliminate the strong coupling singularity both in (4.31), and more generally in all vacua of the form $\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times S^1 \times \mathcal{N}/\,\mathrm{U}(1)$. $\mathbb{R}_\phi \times S^1$ is an infinite cylinder; the dilaton grows as one goes down the cylinder. One can eliminate the strong coupling region by changing the topology of the cylinder to the semi-infinite cigar, which can be described by CFT on $\mathrm{SL}(2)/\,\mathrm{U}(1)$ [41].

The string coupling on the cigar is bounded and, in principle, one should be able to study the theory using worldsheet methods. Since observables are exponentially supported far from the tip of the cigar, where the space looks like a cylinder, much of our discussion above applies to this geometry. To compute correlation functions it is important to take into account scattering from the tip of the cigar. This can be done using results on CFT on euclidean AdS_3 (the coset $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$) and coset CFT techniques.

For example, string theory on

$$\frac{\mathrm{SL}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \times \mathbb{R}^{5,1} \tag{4.36}$$

is a vacuum with sixteen supercharges which looks asymptotically like (4.31), however, unlike (4.31), it should be a weakly coupled theory.[18] According to [42], (4.36) is related to rotating $NS5$-branes.

Note added. String propagation in the "near-horizon" geometry of CY manifolds with hypersurface singularities was also studied in [43, 19]. The relation of our work to these papers is discussed in [44].

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[18] The symmetry structure of this vacuum is discussed in [10, appendix B].

A. Interpolating between linear dilaton and AdS_3 vacua

In the text we mentioned the fact that the two dimensional linear dilaton $\mathbb{R}^{1,1} \times \mathbb{R}_\phi$ and AdS_3 vacua are closely related. We also mentioned that there are solutions which interpolate between the two. In this appendix we construct such solutions. While the construction is general, we present it for the special case of vacua of the form

$$\mathcal{M}_3 \times S^3 \times T^4 \,, \tag{A.1}$$

where $\mathcal{M}_3$ interpolates between $\mathbb{R}^{1,1} \times \mathbb{R}_\phi$ for $\phi \to \infty$ and AdS_3 for $\phi \to -\infty$. We will first describe the solution in supergravity, and then the corresponding exact worldsheet CFT.

Consider a configuration of k $NS5$-branes wrapped on a four-torus of volume vl_s^4, parametrized by the coordinates x_i, $i = 1, 2, 3, 4$. In the remaining non-compact dimensions the fivebranes look like k strings whose worldsheet is the $(\gamma, \bar\gamma)$ plane. One can add to this configuration p fundamental strings parallel to the fivebranes (i.e. extended in $(\gamma, \bar\gamma)$ as well). The metric, dilaton and NS $B_{\mu\nu}$ field for this configuration of branes, with the p strings smeared over the four-torus, are [2, 45, 46]:

$$\begin{aligned} ds^2 &= f_1^{-1} l_s^2 d\gamma d\bar\gamma + f_5(dr^2 + r^2 d\Omega_3^2) + dx_i dx^i \,, \\ e^{2\Phi} &= g_s^2 f_5 f_1^{-1} \,, \\ dB &= \frac{2ipg_s^2}{v} f_5 f_1^{-1} *_6 \epsilon_3' + 2ik\epsilon_3' \,, \end{aligned} \tag{A.2}$$

where $d\Omega_3^2$ and ϵ_3' are the metric and volume form on the unit 3-sphere, $*_6$ is the Hodge dual in the six dimensions parametrized by $(\gamma, \bar\gamma, r, \Omega_3)$ and

$$f_j = 1 + \frac{R_j^2}{r^2} \,, \qquad R_1^2 = \frac{pg_s^2 l_s^2}{v} \,, \qquad R_5^2 = kl_s^2 \,. \tag{A.3}$$

At weak coupling the fivebranes are much heavier than the strings and thus they give rise to a larger distortion of the geometry around them (i.e. typically $R_1 \ll R_5$). Therefore, it makes sense to study an intermediate region in the background (A.2) where one is in the near-horizon geometry of the fivebranes but not necessarily of the strings. As is clear from (A.3), this is the region $r \ll R_5$. In this limit, the geometry has the form (A.1), where the three dimensional manifold $\mathcal{M}_3$ is described by:

$$\begin{aligned} ds_3^2 &= f_1^{-1} l_s^2 d\gamma d\bar\gamma + R_5^2 d\phi^2 \,, \\ e^{2\Phi} &= \frac{v}{p} e^{-2\phi} f_1^{-1} \,, \\ dB &= 2ie^{-2\phi} f_1^{-1} \epsilon_3 = d\left[i f_1^{-1} d\gamma \wedge d\bar\gamma \right] \,, \end{aligned} \tag{A.4}$$

where

$$f_1 = 1 + \frac{1}{k} e^{-2\phi} \,, \qquad e^\phi = \frac{l_s r}{R_1 R_5} = \sqrt{\frac{v}{pk}} \frac{r}{g_s l_s} \,, \tag{A.5}$$

and $\epsilon_3 = *_6\epsilon_3' = \frac{1}{k} f_1^{-1} d\gamma \wedge d\bar\gamma \wedge d\phi$ is the volume form defined by ds_3^2/R_5^2. For $R_1 \ll r$, $f_1 \approx 1$ (A.3) and $\mathcal{M}_3$ looks like flat space with a linear dilaton, $\mathbb{R}^{1,1} \times \mathbb{R}_\phi$. For $r \ll R_1$ one is in the near-horizon geometry of both the strings and the fivebranes and $\mathcal{M}_3$ becomes the familiar AdS_3 solution:

$$\begin{aligned} ds_3^2 &= k l_s^2 (e^{2\phi} d\gamma d\bar\gamma + d\phi^2)\,, \\ e^{2\Phi} &= \frac{kv}{p}\,, \\ dB &= 2ik\epsilon_3 = d\left[ike^{2\phi} d\gamma \wedge d\bar\gamma\right]. \end{aligned} \tag{A.6}$$

Therefore, (A.1) interpolates between the linear dilaton and AdS_3 vacua.

While the above discussion took place in supergravity, one can show that the background (A.4) in fact gives rise to an exact solution of the (classical) string equations of motion, i.e. to a worldsheet CFT with the right properties. This CFT can be constructed as a perturbation of CFT on AdS_3. We next briefly review that construction.

CFT on AdS_3 has an SL(2) $\times$ SL(2) current algebra. One of the null holomorphic currents, J, and its (anti-holomorphic) complex conjugate, $\bar J$, can be written semiclassically as

$$J \sim e^{2\phi}\partial\gamma\,, \qquad \bar J \sim e^{2\phi}\bar\partial\bar\gamma\,. \tag{A.7}$$

The operator $J\bar J$ is exactly marginal in CFT on AdS_3. Adding this perturbation to the AdS_3 background with a finite coefficient, along the lines of [47, 48], gives rise to a one parameter family of sigma models with lagrangean [49]

$$\frac{1}{2\pi}\left(k\partial\phi\bar\partial\phi + \frac{k\alpha}{1+\alpha e^{-2\phi}}\partial\gamma\bar\partial\bar\gamma\right) + \frac{1}{8\pi}\sqrt{g}R^{(2)}\left\{\log[e^{-2\phi}(1+\alpha e^{-2\phi})^{-1}] + \text{const.}\right\}. \tag{A.8}$$

For $\alpha = 1/k$ the sigma model (A.8) is identical to the background (A.4), (A.5). Thus, the background $\mathcal{M}_3$ corresponding to (A.4) is an exact CFT with $c = 3 + 6/k$.

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